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1982 J. Phys. A: Math. Gen. 15 L749

(http://iopscience.iop.org/0305-4470/15/12/016)

J. Phys. A: Math. Gen. 15 (1982) L749–L752. Printed in Great Britain

LETTER TO THE EDITOR

Weak singularity and absence of metastability in random Ising ferromagnets

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Received 17 August 1982

Abstract. In the above models there exists a weak singularity of the magnetisation, in addition to its jump, on the first-order transition line. It corresponds to the Griffiths singularity above the temperature of the phase transition. Analytic continuation from positive to negative magnetic fields, yielding a metastable state, is impossible.

The van der Waals and mean-field theories (see e.g. Brout 1965), in particular the analysis of the positive solution $M(h)$ of the mean-field equation $M = \tanh\,(aM+h)$ for $a>1$ and $h \geqslant 0$ provide examples that thermodynamic functions can be continued analytically to a metastable region. However, this possibility was questioned, on very general grounds, for systems with short-range interactions (Lanford and Ruelle 1969). Droplet-model calculations (Langer 1967, Fisher 1967) exhibit an essential singularity of the free energy as a function of the external field h, on the first-order phase transition line $\{(T, h)\colon T<T_c, h=0\}$ (T denotes the temperature). The singularity in the droplet model permits the analytic continuation *around* the point $h=0$, and the metastable free energy can be interpreted as the real part of the (complex) analytic continuation for $h<0$. The inclusion of 'ramified' clusters (surface per volume $\sim$O(1)) in the droplet model (Domb 1976, Klein 1981) introduces branch point singularities interpreted as spinodals, and may eventually shift the essential singularity to negative external fields (Domb 1976), thus giving space to a 'true' metastable domain.

My aim is to compare the above picture with that inferred rigorously from random dilute Ising ferromagnets. It will be seen that in the latter models (i) there is a weak (i.e. C^{∞}) singularity on the first-order transition line which blocks the analytic continuation *through* $h=0$, and (ii) most probably analytic continuation *around* $h=0$ is impossible or if it is possible then it leads to the stable equilibrium state. These findings agree with the expectation that metastability can be induced only by confining the system in a part of the phase space (Penrose and Lebowitz 1971); indeed, this is implicit in the droplet model but not in our case.

The model I am considering is defined by the Hamiltonian

$$H = -\sum J_{ij} S_i S_j \tag{1}$$

where the summation goes over the nearest-neighbour pairs of a $d \geqslant 2$ dimensional lattice and J_{ij} are random ferromagnetic couplings. In the case of site dilution, $J_{ij} = \sigma_i \sigma_j$ where $\sigma_i = 1$ or 0 and prob $(\sigma_i = 1) = p$, independently of σ_k, $k \neq i$. In the bond diluted model, $J_{ij} = 1$ or 0 with probability p and $1-p$, respectively, and independently of the

0305-4470/82/120749+04$02.00

other bonds. Above one dimension and for p sufficiently close to 1, there is a critical temperature $T_c(p) > 0$ below which magnetic ordering appears.

Recently Georgii (1981) has shown that in the square lattice model $T_c(p)$ is positive for any p above the percolation threshold $p_s \approx 0.590$ (site) or $p_b = 1/2$ (bond); $T_c(p)$ is positive also in higher dimensions if p is greater than the corresponding two-dimensional critical probability, p_s or p_b. Griffiths (1969) revealed that the magnetisation is not an analytic function of h at $h = 0$, at any temperature below $T_c(1)$ and for any $0 < p < 1$; he gave the proof of this statement for $p < p_c(d)$ (the critical probability for percolation). It is obvious that at $T < T_c(p)$ the magnetisation is non-analytic in h at $h = 0$ because it has a jump. A mere jump would still permit the analytic continuation through $h = 0$. As we see, however, this is not the case.

Theorem. Consider the site-diluted model (1) in an external field h, i.e.

$$H = -\sum \sigma_i \sigma_j S_i S_j - h \sum \sigma_i S_i. \tag{2}$$

For any $0 < p < 1$ and $T < T_c(1)$ the quenched magnetisation per site (and therefore the quenched free energy) cannot be continued analytically from $h > 0$ to $h < 0$ and vice versa, through $h = 0$; although there is some $p_0 \geqslant p_s$ and for each $p > p_0$ some $T_0(p) \leqslant T_c(p)$ such that for $p > p_0$ and $T < T_0(p)$ all the one sided derivatives, at $h = 0$, of the magnetisation exist.

Proof. Let Λ be a regular (cubic) domain of the lattice. The quenched magnetisation per site is the field derivative of the quenched free energy and it reads as

$$m_\Lambda = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \overline{\sigma_i \langle S_i \rangle_\Lambda (\sigma_k \sigma_l / T)}. \tag{3}$$

The bar in this formula indicates averaging for all σ_i with i taken in Λ, and $\langle S_i \rangle_\Lambda$ is the expectation value of S_i for a given set σ. Here we assume the free boundary condition ($\sigma_i = 0$ for i outside Λ) and $|\Lambda|$ is the number of sites in Λ. A connected set is an ensemble of sites which are linked together through the edges between the nearest-neighbour pairs within this set; a set C of sites is a cluster if it is connected, fully occupied (i.e. $\sigma_i = 1$ for i in C) and is not connected to any other occupied sites (i.e. $\sigma_i = 0$ for i in ∂C, the set of nearest neighbours to C). With these definitions,

$$m_\Lambda = \sum_{C \subset \Lambda}{}' P_{C,\Lambda} |C| M_C / |\Lambda| \tag{4}$$

where the summation goes over all translationally non-equivalent connected sets of Λ, M_C is the average magnetisation per site in C taken with free boundary condition and $P_{C,\Lambda}$ is the probability that Λ contains a cluster translationally equivalent to C. Now let $z = \exp(-2h/T)$; the interaction (i.e. temperature) dependence of $M_C(z)$ can be expressed through the zeros, $\xi_{i,C}$, of the cluster partition function $Z_C(z)$. Now $|\xi_{i,C}| = 1$ (Lee and Yang 1952) and m_Λ is given as

$$m_\Lambda = p - 2z f_\Lambda(z) \tag{5}$$

with

$$f_\Lambda(z) = \frac{1}{|\Lambda|} \sum_{C \subset \Lambda}{}' P_{C,\Lambda} \sum_{i=1}^{|C|} \frac{1}{z - \xi_{i,C}} = \sum_{j=1}^{N_\Lambda} \frac{\eta_j(\Lambda)}{z - \xi_j}. \tag{6}$$

Here N_Λ is the number of the possible zeros and $\eta_j(\Lambda) = P_{C,\Lambda}/|\Lambda|$ if $\xi_j = \xi_{i,C}$ for some i. In (5) we used the fact that $\Sigma'_{C\subset\Lambda} P_{C,\Lambda}|C|/|\Lambda| = \Sigma_{j=1}^{N_\Lambda} \eta_j(\Lambda) = p$. This and the existence of $\lim_\Lambda m_\Lambda$ for real h ensures that $\lim_\Lambda f_\Lambda(z)$ defines an analytic function $f(z)$ for $|z| \neq 1$. We show that $f(z)$ cannot be continued analytically on the real axis from $z > 1$ to $z < 1$ if $T < T_c(1)$. For, supposing the opposite, there would be some $x > 1$ and $\delta > 0$ so that the disc of convergence of the Taylor series of $f(z)$ about $z = x$ would contain an arc, $A = \{\exp(i\varphi): |\varphi| < \delta\}$ in its interior. Now T is below the critical temperature of the non-diluted model, therefore on an increasing sequence of fully occupied cubes C_n the sets of zeros $\{\xi_{i,C_n}\}$ accumulate to $z = 1$ as n goes to infinity. Let C_m be so large that $\xi_{j,C_m} \in A$ for some j. For any $\Lambda \supset C_m$, $\xi_{j,C_m} \equiv \xi_k$ will be found among the poles of $f_\Lambda(z)$. It is seen that $P_{C,\Lambda}/|\Lambda|$ tends to $P_C = p^{|C|}(1-p)^{|\partial C|}$ as Λ tends to the infinite lattice; in particular, there is some $\Lambda_0 \supset C_m$ so that $\eta_k(\Lambda) \geq P_{C_m}/2$ if $\Lambda \supset \Lambda_0$. Let now $\xi_k = \exp(i\varphi_k)$ and, for $r > 1$ real, $r\xi_k - \xi_j = r_{kj}\exp(i\varphi_{kj})$ where $r_{kj} = |r\xi_k - \xi_j| > 0$. From the convexity of the unit circle it follows that $|\varphi_k - \varphi_{kj}| < \frac{1}{2}\pi$ and hence

$$|f_\Lambda(r\xi_k)| \geq \mathrm{Re}(f_\Lambda(r\xi_k)\xi_k) = \frac{\eta_k(\Lambda)}{r-1} + \sum_{j\neq k} \frac{\eta_j(\Lambda)}{r_{kj}} \cos(\varphi_k - \varphi_{kj}) \geq \frac{\eta_k(\Lambda)}{r-1} \geq \frac{1}{2}\frac{P_{C_m}}{r-1} \tag{7}$$

for any $\Lambda \supset \Lambda_0$. This shows that $f(r\xi_k)$ diverges if r decreases to 1, contradicting the supposed analyticity on A. A similar divergence can be found if $r < 1$ increases to 1 as a consequence of the equality $m_\Lambda(z) = -m_\Lambda(z^{-1})$.

As to the second part of the theorem, for $p > p_0(d) = (1 - \frac{1}{3}^{2d})^{1/2}$ and for T low enough one can show, by an 'averaged' Peierls argument, the exponential decay of the truncated correlations and, following Martin-Löf (1973), the stated C^∞ property.

A few comments are necessary to this result. It constitutes a re-derivation of the Griffiths singularity if $p < p_c(d)$ and $T_c(p) < T < T_c(1)$. Equation (6) is a close analogue of the formula (3) of Griffiths (1969). The difference is that the infinite clusters are included in the present treatment: they occur for $p > p_c(d)$. The domain $p > p_c(d)$, $T_c(p) < T < T_c(1)$ is still that of the Griffiths singularity; the singularity, superposed on the jump of the magnetisation, appears below $T_c(p)$. We cannot separate the jump and the excess singularity: it is probably not true that only the finite clusters provide the essential singularity and the infinite clusters add a pure jump to it (this would mean metastability in the non-diluted model). The estimated domain where the singularity is of C^∞, is rather poor. For two dimensions, $p_0(2) = 0.9938$. It seems probable that the infinite differentiability is true for the whole domain of p and T where the singularity occurs. The bond problem can be treated quite analogously; in this case we obtain $p_0(d) = \frac{2}{3}$.

Chances for the analytic continuation *around* $h = 0$ (i.e. $z = 1$) are weak. A mean-field calculation and series expansions show (Kortman and Griffiths 1971) that the zeros are everywhere dense on the unit circle, therefore the same kind of singularity can be found in each point of this circle. Assuming nevertheless that there is an arc free of zeros, the analytic continuation through this arc from a $z_1 = \mathrm{Re}\, z_1 > 1$ to a $z_2 = \mathrm{Re}\, z_2 < 1$ would inevitably yield the stable equilibrium magnetisation at z_2, instead of a metastable one.

I would like to thank J-J Loeffel, H Kunz and Ch Pfister for many enlightening conversations on the Griffiths singularity, and R B Griffiths for a correspondence on this subject. I am indebted to W Klein who directed my attention to the problem of the weak singularity in first-order phase transitions. I also would like to thank the Institut de Physique Théorique, Université de Lausanne for the kind hospitality I enjoyed there during my stay in Lausanne.

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